Derivation of the Schawlow–Townes Linewidth of Lasers

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Content

(see also: article on the Schawlow–Townes linewidth in the Encyclopedia of Laser Physics and Technology)

1 History

Original paper: A. L. Schawlow and C. H. Townes, "Infrared and optical masers", Phys. Rev. 112 (6), 1940 (1958):

- Basic proposal for building lasers (optical masers)
- Estimate of linewidth in analogy to linewidth of masers. The latter is limited by thermal fluctuations; Schawlow and Townes basically replaced $k_B T$ with $h\nu$ in the linewidth equation. The result is

$$\Delta\nu_{\text{laser}} = \frac{4\pi\, h\nu\, (\Delta\nu_c)^2}{P_{\text{out}}} \tag{1}$$

  where $\Delta\nu_{\text{laser}}$ is the *half* width at half-maximum linewidth of the laser, $\Delta\nu_c$ is the *half* width of the resonances of the laser resonator and P_{out} is the laser output power. Note that this holds only if there are no parasitic resonator losses (see section 3).

- In 1967, Melvin Lax showed that the linewidth in lasing operation (above threshold) must be two times smaller. By taking into account this correction and also converting the equation for *full* width at half maximum (for both linewidth quantities), we arrive at

$$\Delta\nu_{\text{laser}} = \frac{\pi\, h\nu\, (\Delta\nu_{\text{c}})^2}{P_{\text{out}}} \tag{2}$$

As the derivation by Schawlow and Townes largely builds on previous results, which are not well available, it is desirable to have a more comprehensive and complete derivation, which can also generalize the results.

2 Physical Interpretation

The basic physical process which limits the linewidth is spontaneous emission into the laser mode:

- In each round-trip, some noise amplitude is added to the circulating field. This changes both amplitude (power) and phase of the field.
- Amplitude fluctuations are damped: because of gain saturation, the power must always return to values close to the steady-state power.
- For phase fluctuations, there is no restoring force. Therefore, the phase undergoes a random walk. This leads to a phase noise power spectral density $S_\varphi(f) \propto f^{-2}$, which causes a finite linewidth.

Remarks:

- Depending on the type of semiclassical picture used, there can also be fluctuations associated with the resonator losses.
- Eq. (1) holds only for four-level single-frequency lasers without amplitude/phase coupling and with negligible other noise sources (such as mirror vibrations).
- Most lasers operate far above the Schawlow-Townes limit, as there are additional technical noise sources.
- For semiconductor lasers, a factor $(1+\alpha^2)$ (with α being the linewidth enhancement factor of the gain medium) must be added. This is due to amplitude/phase coupling in the semiconductor: changes of gain are accompanied by changes of refractive index.

3 Derivation of Linewidth Formula

In the following, I give a derivation which does not build on early linewidth formulas for (thermally limited) masers, but rather on formulae for quantum noise which I have also used for numerical simulations of timing jitter of mode-locked lasers.

Assume a four-level single-frequency laser without amplitude/phase coupling. The output coupler transmission is T_{oc}, and there can be additional (parasitic) resonator losses l_{par}, so that the total resonator losses per round trip are $l_{\text{tot}} = T_{\text{oc}} + l_{\text{par}}$. On average, the gain must balance the losses: $g = l_{\text{tot}}$.

We describe the circulating field with a complex amplitude A, normalized so that the intracavity power is $P_{\text{int}} = |A|^2$.

During each resonator round-trip, the gain medium adds a fluctuating amplitude δA where each quadrature component has the variance

$$\sigma^2 = \frac{h\nu}{4 \cdot T_{\mathrm{rt}}} g \tag{3}$$

(according to some basic results of quantum optics) with T_{rt} being the round-trip time.

This means that the phase changes by

$$\delta\varphi = \frac{\delta A_{\mathrm{q}}}{A} \tag{4}$$

where δA_{q} is the quadrature component perpendicular to A in the complex plane.

The resonator losses contribute another noise amplitude with the same variance (and with no correlation to the fluctuation induced by the gain), because $g = l_{\mathrm{tot}}$.

Therefore, in total the variance of the phase is increased by

$$\delta\sigma_\varphi^2 = \frac{h\nu}{4 \cdot T_{\mathrm{rt}} \cdot |A|^2} g + \frac{h\nu}{4 \cdot T_{\mathrm{rt}} \cdot |A|^2} l_{\mathrm{tot}} = \frac{h\nu}{2 \cdot T_{\mathrm{rt}} \cdot P_{\mathrm{int}}} l_{\mathrm{tot}} \tag{5}$$

per round-trip. (We have used the average value of the intracavity power in the dominator, which is a good approximation.)

Therefore, the phase variance grows with time according to

$$\sigma_\varphi^2(t) \equiv \left\langle \left(\varphi(t) - \varphi(0)\right)^2 \right\rangle = \frac{h\nu}{2 \cdot T_{\mathrm{rt}} \cdot P_{\mathrm{int}}} l_{\mathrm{tot}} \frac{t}{T_{\mathrm{rt}}} = \frac{h\nu\, l_{\mathrm{tot}}}{2 \cdot T_{\mathrm{rt}}^2 \cdot P_{\mathrm{int}}} t\ . \tag{6}$$

It can be shown that a random-walk process with the variance

$$\sigma_\varphi^2(t) = 2\pi\, \tilde{C}\, t\ , \tag{7}$$

corresponds to a power spectral density of the phase noise of

$$S_\varphi(\omega) = \frac{\tilde{C}}{\omega^2}\ , \tag{8}$$

and this results in the linewidth

$$\Delta\nu = \tilde{C} \tag{9}$$

of the field. From this, we obtain the laser linewidth

$$\Delta\nu_{\mathrm{laser}} = \frac{\sigma_\varphi^2(t)}{2\pi\, t} = \frac{h\nu\, l_{\mathrm{tot}}}{4\pi\, T_{\mathrm{rt}}^2\ P_{\mathrm{int}}} = \frac{h\nu\, l_{\mathrm{tot}} T_{\mathrm{oc}}}{4\pi\, T_{\mathrm{rt}}^2\ P_{\mathrm{out}}}\ . \tag{10}$$

so that

$$\Delta\nu_{\mathrm{laser}} = \frac{h\nu\, l_{\mathrm{tot}} T_{\mathrm{oc}}}{4\pi\, T_{\mathrm{rt}}^2\ P_{\mathrm{out}}}\ . \tag{11}$$

Although eq. (11) appears to be most practical, for comparison with eq. (1) we relate the resonator losses to the FWHM resonator bandwidth:

$$\Delta \nu_{\mathrm{c}} = \frac{1}{2\pi} \Delta\omega_{\mathrm{c}} = \frac{1}{2\pi} \frac{l_{\mathrm{tot}}}{T_{\mathrm{rt}}} \tag{12}$$

If there are no parasitic losses, so that $l_{\mathrm{tot}} = T_{\mathrm{oc}}$, this leads to eq. (2). We thus see that the result of the derivation is consistent with the Schawlow–Townes result as modified by Lax.

Another useful result is that the coherence time is

$$\tau_{\mathrm{coh}} = \frac{1}{\pi \cdot \Delta \nu_{\mathrm{laser}}} = \frac{P_{\mathrm{out}}}{\pi^2 \, h\nu \left(\Delta \nu_{\mathrm{c}}\right)^2} \, . \tag{13}$$

This can be used to calculate the autocorrelation of the field:

$$\left\langle A^*(t) A(0) \right\rangle = P_{\mathrm{int}} \, \exp\left(-\frac{|t|}{\tau_{\mathrm{coh}}} \right). \tag{14}$$

4 Extensions

4.1 Three-level Laser

In a three-level laser, there is reabsorption from the ground state population. To obtain the same effective gain, we need a correspondingly higher upper-level population. As a consequence, we get more noise from spontaneous emission and also noise from the reabsorption.

We can describe the gain as

$$g = g_0 - l_{\mathrm{reabs}} \tag{15}$$

where l_{reabs} is the reabsorption loss in the lasing state and g_0 the gain which we would have for this inversion if there were no reabsorption.

Compared with the situation in a four-level laser with the same gain, the noise contribution is increased by the "spontaneous emission factor"

$$n_{\mathrm{sp}} = \frac{g_0 + \left(l_{\mathrm{tot}} + l_{\mathrm{reabs}}\right)}{g + l_{\mathrm{tot}}} = \frac{\left(g + l_{\mathrm{reabs}}\right) + \left(g + l_{\mathrm{reabs}}\right)}{2g} = 1 + \frac{l_{\mathrm{reabs}}}{g} \, . \tag{16}$$

(Note that the total resonator losses l_{tot} do *not* include the reabsorption loss.)

For example, we have $n_{\mathrm{sp}} = 2$ if $l_{\mathrm{reabs}} = g = l_{\mathrm{tot}}$, so that g_0 gets twice as high as it would be without the reabsorption effect.

Note that l_{reabs} is smaller than the unpumped reabsorption loss $l_{\mathrm{reabs,0}}$, because the ground state population of the laser-active ions is reduced. It can be calculated from

$$l_{\mathrm{reabs}} = \frac{N_1}{N_{\mathrm{tot}}} l_{\mathrm{reabs,0}} = \frac{1}{N_{\mathrm{tot}}} l_{\mathrm{reabs,0}} \frac{N_{\mathrm{tot}} \sigma_{\mathrm{em}} - g / L}{\sigma_{\mathrm{em}} + \sigma_{\mathrm{abs}}} = l_{\mathrm{reabs,0}} \frac{\sigma_{\mathrm{em}} - g / L N_{\mathrm{tot}}}{\sigma_{\mathrm{em}} + \sigma_{\mathrm{abs}}} \tag{17}$$

where N_1 is the ground state population density and N_{tot} is the doping density.

If we call the fully inverted gain $g_{\mathrm{fi}} = N_{\mathrm{tot}} \sigma_{\mathrm{em}} L$, this results in

$$l_{\mathrm{reabs}} = l_{\mathrm{reabs,0}} \frac{1 - g / g_{\mathrm{fi}}}{1 + \sigma_{\mathrm{abs}} / \sigma_{\mathrm{em}}} . \tag{18}$$

4.2 Laser with Amplitude/Phase Coupling

Particularly from semiconductor lasers it is known that a change of intensity gain coefficient g is related to a change of phase via the linewidth enhancement factor α :

$$\Delta \varphi = -\frac{\alpha}{2} \Delta g \tag{19}$$

This leads to a coupling of amplitude and phase fluctuations. In effect, the power spectral density of the phase fluctuations in a single-frequency laser is increased by the factor $1+\alpha^2$ (see e.g. C. H. Henry, IEEE J. Quantum Electron. 18 (2), 259 (1982)). The linewidth is increased by the same factor and becomes

$$\Delta \nu_{\mathrm{laser}} = \frac{h\nu \, (1+\alpha^2) \, l_{\mathrm{tot}} T_{\mathrm{oc}}}{4\pi \, T_{\mathrm{rt}}^2 \, P_{\mathrm{out}}} . \tag{20}$$

Note that solid-state lasers can also have a significant linewidth enhancement factor, e.g. when the gain line is asymmetric or it is a quasi-three-level system.

In the following, I give a new derivation of this result, because I find Henry's derivation somewhat unclear. The basic idea of my derivation is:

- Quantum noise from spontaneous emission and losses leads to fluctuations of optical power.
- For low noise frequencies, the gain directly follows fluctuations of the optical power. I do *not* directly base the argument on gain saturation, but rather consider the addition of fluctuations to the field as a kind of gain, which (for low frequencies) must be compensated by opposite changes of the laser gain.
- White noise in the field fluctuations leads to white noise of the gain, consequently to f^{-2} noise in the optical phase, from which I calculate the contribution to the linewidth.

Now the details. Any fluctuation δA in the amplitude quadrature amounts to a change

$$\delta P_{\mathrm{int}} = 2A \, \delta A = \left(\frac{2}{\sqrt{P_{\mathrm{int}}}} \, \delta A \right) P_{\mathrm{int}} \tag{21}$$

of the intracavity optical power. Written in this way, one can consider the addition of the fluctuation amplitude as a kind of gain, which for low frequencies must be compensated by the opposite change of the laser gain, caused by gain saturation:

$$\delta g = -\frac{2}{\sqrt{P_{\mathrm{int}}}} \, \delta A \tag{22}$$

The optical amplitudes have white noise with the two-sided power spectral density

$$S_{\delta A}(f) = \frac{h\nu}{2} g \tag{23}$$

if the effects of quantum fluctuations from gain medium and losses are added. Therefore, we have

$$S_{\delta g}(f) = \frac{4}{P_{\text{int}}} S_{\delta A}(f) = \frac{2h\nu}{P_{\text{int}}} g \,, \tag{24}$$

which indicates white noise in the gain. For the instantaneous optical frequency ν_{opt} and the optical phase φ we have the dynamical equation

$$2\pi\, \delta\nu_{\text{opt}} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi = \frac{\alpha}{2T_{\text{rt}}}\delta g \,, \tag{25}$$

because the optical phase changes by $\alpha\, \delta g / 2$ per round trip. From this we obtain the power spectral density

$$S_{\nu,\text{opt}}(f) = \frac{\alpha^2}{16\pi^2 T_{\text{rt}}^2} S_{\delta g}(f) = \frac{\alpha^2\, h\nu\, g}{8\pi^2 T_{\text{rt}}^2 P_{\text{int}}} \tag{26}$$

of the optical frequency and

$$S_{\varphi}(f) = S_{\nu,\text{opt}}(f) / f^2 = \frac{\alpha^2\, h\nu\, g}{8\pi^2 T_{\text{rt}}^2 P_{\text{int}}} f^{-2} \tag{27}$$

of the optical phase, from which we obtain the contribution to the linewidth in the same way as in section 3 as

$$\Delta\nu = 2\pi \frac{\alpha^2\, h\nu\, g}{8\pi^2 T_{\text{rt}}^2 P_{\text{int}}} = \frac{\alpha^2\, h\nu\, g}{4\pi\, T_{\text{rt}}^2\, P_{\text{int}}} = \alpha^2 \frac{h\nu\, l_{\text{tot}} T_{\text{oc}}}{4\pi\, T_{\text{rt}}^2\, P_{\text{out}}} \,. \tag{28}$$

Finally, one has to add this contribution to that one of the direct influence of quantum noise on the phase fluctuations, since both fluctuations are from different quadrature components and thus statistically independent. In total, one obtains

$$\Delta\nu = 2\pi \frac{\alpha^2\, h\nu\, g}{8\pi^2 T_{\text{rt}}^2 P_{\text{int}}} = \frac{\alpha^2\, h\nu\, g}{4\pi\, T_{\text{rt}}^2\, P_{\text{int}}} = \left(1+\alpha^2\right) \frac{h\nu\, l_{\text{tot}} T_{\text{oc}}}{4\pi\, T_{\text{rt}}^2\, P_{\text{out}}} \tag{29}$$

which is consistent with Henry’s result.