Relaxation Oscillations

Dynamical equations for the *intracavity* power P and the gain g:

$$\frac{\partial}{\partial t} P = \frac{1}{T_{\mathrm{R}}}\left(g - l\right) P \tag{1}$$

$$\frac{\partial}{\partial t} g = -\frac{g - g_0(P_{\mathrm{p}})}{\tau_2} - \frac{gP}{E_{\mathrm{sat}}} \tag{2}$$

where T_{R} is the cavity roundtrip time, l is the total cavity loss (output coupler transmission + parasitic losses), $g_0(P_{\mathrm{p}})$ is the small-signal gain (dependent on the pump power P_{p}), τ_2 is the upper-state lifetime, and E_{sat} is the saturation energy of the gain medium.

These equations hold for a 4-level or quasi-3-level system. The quasi-3-level nature only affects the form of the function $g_0(P_{\mathrm{p}})$: we then have $g_0(0) < 0$ due to reabsorption losses. The saturation energy then has to be defined with the sum of emission and absorption cross sections at the laser wavelength.

We linearize the dynamical equations for small deviations from the steady state:

$$\frac{\partial}{\partial t} \delta P = \frac{1}{T_{\mathrm{R}}}\left(g - l\right) \delta P + \frac{P}{T_{\mathrm{R}}} \delta g \tag{3}$$

$$\frac{\partial}{\partial t} \delta g = -\left(\frac{1}{\tau_2} + \frac{P}{E_{\mathrm{sat}}}\right) \delta g - \frac{g}{E_{\mathrm{sat}}} \delta P \tag{4}$$

where $g - l = 0$ in the first equation. Combining both equations we obtain

$$\frac{\partial^2}{\partial t^2} \delta g + \left(\frac{1}{\tau_2} + \frac{P}{E_{\mathrm{sat}}}\right) \frac{\partial}{\partial t} \delta g + \frac{gP}{T_{\mathrm{R}} E_{\mathrm{sat}}} \delta g = 0 . \tag{5}$$

With the ansatz $g \propto \exp(st)$ (with complex s) we obtain

$$s^2 + \left(\frac{1}{\tau_2} + \frac{P}{E_{\mathrm{sat}}}\right) s + \frac{gP}{T_{\mathrm{R}} E_{\mathrm{sat}}} = 0 \tag{6}$$

and therefore

$$s_{1,2} = -\frac{1}{2\tau_{\mathrm{eff}}} \pm \sqrt{\frac{1}{4\tau_{\mathrm{eff}}{}^2} - \frac{gP}{T_{\mathrm{R}} E_{\mathrm{sat}}}} \tag{7}$$

where

$$\frac{1}{\tau_{\mathrm{eff}}} \equiv \frac{1}{\tau_2} + \frac{P}{E_{\mathrm{sat}}} . \tag{8}$$

For solid-state lasers, we normally have $\frac{1}{4\tau_{\text{eff}}^{\;2}} << \frac{gP}{T_{\text{R}} E_{\text{sat}}}$ (class-B regime) (except for operation close to threshold), so that

$$s_{1,2} \approx -\frac{1}{2\tau_{\text{eff}}} \pm i\sqrt{\frac{gP}{T_{\text{R}} E_{\text{sat}}}} \, . \tag{9}$$

Also we can use $g = l$.

From this we learn that the relaxation oscillations have the frequency

$$f_{\text{ro}} = \frac{1}{2\pi}\sqrt{\frac{l\,P}{T_{\text{R}} E_{\text{sat}}}} \tag{10}$$

and the damping time

$$\tau_{\text{ro}} = 2\tau_{\text{eff}} = \frac{2}{\frac{1}{\tau_2} + \frac{P}{E_{\text{sat}}}} \, . \tag{11}$$

Close to threshold, we have $\tau_{\text{ro}} \approx 2\tau_2$. For higher pump powers, the damping becomes stronger.

Some authors replace l / T_{R} with the cavity damping rate τ_{c}^{-1}.

If parasitic losses are negligible, $l\,P$ is the output power. Otherwise we may write $l\,P = P_{\text{out}} / \eta_{\text{oc}}$ with the output coupling efficiency $\eta_{\text{oc}} = \frac{T_{\text{oc}}}{T_{\text{oc}} + l_{\text{par}}}$.

Only for 4-level lasers, we know that the small-signal gain is proportional to the pump power, and

$$\frac{1}{\tau_{\text{eff}}} \equiv \frac{1}{\tau_2} + \frac{P}{E_{\text{sat}}} = \frac{r}{\tau_2} \tag{12}$$

where r is the ratio of pump power and threshold pump power, so that we have

$$f_{\text{ro}} \approx \frac{1}{2\pi}\sqrt{\frac{l\,P}{E_{\text{sat}} T_{\text{R}}}} = \frac{1}{2\pi}\sqrt{\frac{l}{T_{\text{R}}}\frac{r-1}{\tau_2}} = \frac{1}{2\pi}\sqrt{\frac{r-1}{\tau_{\text{c}}\tau_2}} \tag{13}$$

and

$$\tau_{\text{ro}} \approx 2\tau_{\text{eff}} = 2\tau_2 / r \, . \tag{14}$$